performed, and the results are presented in Figs. 4 and 5. For CO_2 at 2500°F and 1 atm, it is found, by comparing Fig. 4 with corresponding ϵ_λ results (Fig. 2), that the effective wavelength-independent coefficient A is about 0.05 for all four values of H. For H_2O at 2000°R and 1 atm, the value of A is about 0.25 for different values of H.

Formulation for calculating the apparent emissivity of conical gray-gas bodies has been given in Ref. 2, in which numerical results were obtained for conical gas bodies of apex angles 20°, 60°, and 120°.

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Effect of Aerodynamic Drag on Low-Thrust Ascending-Spiral Trajectories

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Nomenclature

a = acceleration
A = reference area of satellite
$\bar{c}$ = average exhaust velocity of jet
C_D = drag coefficient (drag force)/$\frac{1}{2}\rho v^2 A$
E = total energy per unit mass
g = gravitational acceleration
k = constant; for k_1, k_2, and k_3, see Eqs. (11), (14), and (15), respectively
$\dot{m}$ = constant mass flow rate
M = mass of satellite, $M \equiv M(t) \equiv M_0 - \dot{m}t$
n = number showing a_0 ($= \dot{m}\bar{c}/M_0$) in terms of g_e
S = semimajor axis
t = time
T = actual time required to escape, including effect of drag
v = velocity
β = reciprocal of scale height
$\gamma(a_0)$ = correction factor from Ref. 4
ϵ = eccentricity of an elliptical orbit
θ = polar angle
μ = gravitational constant of earth
ρ = atmospheric density
ω = angular velocity as defined by $\mu^{1/2}/S^{3/2}$

Subscripts

D = drag
e = at earth's surface
E = escape
ED = escape including effect of drag
0 = initial value at $t = 0$
t = thrust

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For low-thrust orbital maneuvers of a low-altitude earth satellite, the perturbation due to aerodynamic forces plays a dominant role in computations of trajectories and propulsion requirements. The case of greatest practical importance in orbital maneuvering is that of tangentially directed thrust, which results not only in the greatest instantaneous rate of energy change, but also in a good approximation to the optimal fuel expenditure.[1] This paper treats ascending spiral trajectories under the influences of constant tangential thrust and aerodynamic drag. The results derived constitute the zero-order approximation to the problem, whereas the first-order approximation will invariably contain oscillatory terms, which, in some similar simpler problems, have been well explored.[2,3]

Analysis

The velocity of a satellite in an elliptical orbit is

$$v = \left[\frac{\mu(1+\epsilon^2)}{S(1-\epsilon^2)}\right]^{1/2}\left[1 + \frac{\epsilon}{1+\epsilon^2}\cos\theta - \frac{1}{2}\left(\frac{\epsilon}{1+\epsilon^2}\right)^2 \times \cos^2\theta + \ldots\right] \quad (1)$$

If the orbit is nearly circular (ϵ is very small), the higher-order terms in the bracketed series may be neglected, and after averaging over one revolution, the result becomes

$$v = \left[\frac{\mu(1+\epsilon^2)}{S(1-\epsilon^2)}\right]^{1/2} \simeq (\mu/S)^{1/2}(1+\epsilon^2) \quad (2)$$

If the orbit remains nearly circular,

$$v = (\mu/S)^{1/2}(1+\epsilon_0^2) \quad (3)$$

Thus with only a small error, of the order of magnitude of ϵ_0^2, the velocity of a low-altitude earth satellite in a nearly circular orbit can be approximated by

$$v = (\mu/S)^{1/2} \quad (4)$$

Under a low constant tangential thrust, its acceleration is

$$a_t = \dot{m}\bar{c}/M \quad (5)$$

its deceleration due to aerodynamic drag is

$$a_D = C_D A\rho v^2/2M \quad (6)$$

and its rate of change of energy is

$$dE/dt = (a_t - a_D)v = (\dot{m}\bar{c} - C_D A\rho v^2/2)v/M \quad (7)$$

in which, for an ascending spiral trajectory, a_t should be greater than a_D, and

$$E = -\mu/2S \qquad \text{or} \qquad dE/dt = (\mu/2S^2)dS/dt \quad (8)$$

Atmospheric density will be approximated by

$$\rho = \rho_0 \exp[-\beta(S - S_0)] = \rho_0 \exp(\beta S_0)\exp(-\beta S) \quad (9)$$

where β is chosen for best fit in the region where the first part of the spiral trajectory (where drag is significant) is located.

Because $(a_t - a_D)$ is small, the orbit remains nearly circular; thus in combining Eqs. (4, 7, 8, and 9), one obtains

$$\frac{\dot{m}\bar{c}}{M}\left[1 - \frac{C_D A\rho_0\mu}{2\dot{m}\bar{c}S}\exp(\beta S_0)\exp(-\beta S)\right] = \frac{\mu^{1/2}}{2S^{3/2}}\frac{dS}{dt} \quad (10)$$

After rearranging various terms and expanding the terms in the brackets into series, the result is

$$-\bar{c}\,dM/M = \tfrac{1}{2}\mu^{1/2}S^{-3/2}\{1 + k_1[S\exp(\beta S)]^{-1} + k_1^2[S\exp(\beta S)]^{-2} + \ldots\}dS \quad (11)$$

where $k_1 \equiv C_D A \rho_0 \mu \exp(\beta S_0)/2\dot{m}\bar{c}$. The infinite series in the brackets converges because $a_D < a_t$. Integrating Eq. (11) with $S = S_0$ at $t = 0$ yields

$$\bar{c}\ln(M_0/M) = \tfrac{1}{2}\mu^{1/2}\left[2(S_0^{-1/2} - S^{-1/2}) + k_1\int_{S_0}^{S} S^{-7/2}\exp(-2\beta S)\,dS + k_1^2\int_{S_0}^{S} S^{-7/2} \times \exp(-2\beta S)dS + \ldots\right] \quad (12)$$

Since βS has the order of magnitude of 10^2, it can be shown that all terms beyond the first in the integrated terms can be neglected, so that

$$\int_{S_0}^{S} S^{-5/2}\exp(-\beta S)dS \simeq \frac{1}{\beta}S_0^{-2}\exp(-\beta S_0) \times \{1 - (S_0/S)^{5/2}\exp[-\beta(S - S_0)]\}$$
$$\int_{S_0}^{S} S^{-7/2}\exp(-2\beta S)dS \simeq \frac{1}{2\beta}S_0^{-3}\exp(-2\beta S_0) \times \{1 - (S_0/S)^{7/2}\exp[-2\beta(S - S_0)]\} \quad (13)$$

With such approximations, Eq. (12) becomes the desired zero-order approximation for the trajectories:

$$\bar{c}\ln(M_0/M) = \tfrac{1}{2}v_0(2[1 - (S_0/S)^{1/2}] - (\beta S_0)^{-1}\ln(1 - k_2 S_0) + (\beta S_0)^{-1}(S_0/S)^3\ln\{1 - k_2(S_0^2/S) \times \exp[-\beta(S - S_0)]\}) \quad (14)$$

where $k_2 \equiv C_D A g_0 \rho_0 / 2M_0 g_e n$. When $S - S_0$ is very small, the third term in parentheses must be retained to obtain the desired accuracy, but when $\beta(S - S_0) = O(10^1)$ or greater, the third term may be neglected, leaving

$$\bar{c}\ln(M_0/M) = v_0[(1 + k_3) - (S_0/S)^{1/2}] \quad (15)$$

where $k_3 \equiv -(2\beta S_0)^{-1}\ln(1 - k_2 S_0)$. For an elliptical orbit the angular momentum relationship is given by

$$d\theta/dt = \mu^{1/2}(1 + \epsilon\cos\theta)/[S(1 - \epsilon^2)]^{3/2} \quad (16)$$

Averaging over a period of revolution gives

$$d\theta/dt = \mu^{1/2}/[S(1 - \epsilon^2)]^{3/2} \quad (17)$$

With zero-order approximation, Eq. (17) becomes

$$\theta = \mu^{1/2}t/[S_0(1 - \epsilon_0^2)]^{3/2} \quad (18)$$

which shows that a good approximation of θ, with an accuracy to the order of magnitude of ϵ_0^2, is

$$\theta = \mu^{1/2}S_0^{-3/2}t = \omega_0 t \quad (19)$$

Since $dS/dt = (dS/d\theta)(d\theta/dt)$, one obtains a result similar to Eq. (11), which, after integrating with the initial condition $\theta = 0$, $S = S_0$, with the approximations used in obtaining Eq. (14), becomes

$$\bar{c}\ln[M_0/(M_0 - \dot{m}\theta/\omega_0)] = \tfrac{1}{2}v_0(\tfrac{1}{2}[1 - (S_0/S)^2] - (\beta S_0)^{-1}\ln(1 - k_2 S_0) + (\beta S_0)^{-1}(S_0/S)^3 \times \ln\{1 - k_2(S_0^2/S)\exp[-\beta(S - S_0)]\}) \quad (20)$$

This is the zero-order approximation for the spiral trajectories expressed in terms of S and θ. Again, when $S - S_0$ is large, Eq. (20) can be simplified to

$$\bar{c}\ln[M_0/(M_0 - \dot{m}\theta/\omega_0)] = \tfrac{1}{4}v_0[(1 + k_3) - (S_0/S)^2] \quad (21)$$

Case of Escape

At escape, S approaches infinity, the last term falls out of Eq. (15), and the time required to escape from the orbit S_0 is

$$t_{ED} = M_0\{1 - \exp[-v_0(1 + k_3)/\bar{c}]\}/\dot{m} \quad (22)$$

If there is no aerodynamic drag, $k_3 = 0$, and

$$t_E = M_0[1 - \exp(-v_0/\bar{c})]/\dot{m} \quad (23)$$

which is a known result shown elsewhere.[3,4]

Since the last few turns of the trajectory actually deviate from circular, t_E calculated by Eq. (23) gives a larger value than the actual time of escape as calculated by numerical integration. In order to obtain the exact time of escape, Melbourne[4] introduced a factor $\gamma(a_0)$ such that $\gamma(a_0)t_E$ will give the correct escape time [see his Fig. 25 for $\gamma(a_0)$ as a function of a_0]. Because of the atmospheric density distribution, most of the additional time owing to aerodynamic drag is spent on the turns near the earth, whereas the factor $\gamma(a_0)$ is used to correct for the noncircularity of the last few turns of the escape trajectory (i.e., the opposite end). Hence, the time spent to counteract the aerodynamic drag is

$$t_D = t_{ED} - t_E = M_0\exp(-v_0/\bar{c}) \times [1 - \exp(-k_3 v_0/\bar{c})]/\dot{m} \quad (24)$$

and the actual escape time is found by

$$T = \gamma(a_0)t_E + t_D \quad (25)$$

The additional fuel required due to drag is

$$M_D = t_D\dot{m} = M_0\exp(-v_0/\bar{c})\,[1 - \exp(-k_3 v_0/\bar{c})] \quad (26)$$

The number of revolutions at escape is determined from Eq. (21) by setting $S = \infty$, which drops the last term, so that the angle traversed is

$$\theta_{ED} = M_0\omega_0\{1 - \exp[-v_0(1 + k_3)/4\bar{c}]\}/\dot{m} \quad (27)$$

If there is no aerodynamic drag, $k_3 = 0$, and

$$\theta_E = \frac{M_0\omega_0}{\dot{m}}\cdot\frac{v_0}{4\bar{c}}\left[1 - \frac{1}{2!}\left(\frac{v_0}{4\bar{c}}\right) + \frac{1}{3!}\left(\frac{v_0}{4\bar{c}}\right)^2 - \ldots\right] \quad (28)$$

As a check of the present analysis for this special case (escape with no drag), Eq. (28) is compared with a known, accurate result by Melbourne[4]:

$$\theta_E = \frac{M_0\omega_0}{\dot{m}}\cdot\frac{v_0}{4\bar{c}}\left[1 - \frac{1}{5}\frac{v_0}{\bar{c}} + \frac{1}{30}\left(\frac{v_0}{\bar{c}}\right)^2 - \frac{1}{210}\left(\frac{v_0}{\bar{c}}\right)^3 + \ldots\right] \quad (29)$$

It is seen that differences in θ for most practical problems are within a few percent.

The additional angle traversed due to drag is

$$\theta_D = \theta_{ED} - \theta_E = (M_0\omega_0/\dot{m})\exp(-v_0/4\bar{c}) \times [1 - \exp(-k_3 v_0/4\bar{c})] \quad (30)$$

Table 1 Effects of drag on ascent and escape from 100-naut-mile orbit

Mission		Time and propellant required			Turns traversed	
		Days	% M_0	Factor	No.	Factor
100–300 naut miles	With drag	13.55	1.20	1.122	190	1.038
	No drag	12.08	1.07	1.0	183	1.0
100 naut miles to escape	With drag	353	31.3	1.006	1762	1.006
	No drag	351	31.1	1.0	1751	1.0

Numerical Example

For a satellite originally in a 100-naut-mile circular orbit with a constant tangential thrust of $2.05 \times 10^{-5} g_e M_0$, the following are assumed:

$\rho_0 = 5 \times 10^{-13}$ slug ft^{-3}

$C_D A/2M_0 = 1$ ft^2 slug^{-1}

$\bar{c} = 64.4 \times 10^3$ fps

$\beta = 4.74 \times 10^{-6}$ ft^{-1}

$\gamma(a_0) = 0.945$ from Ref. 4

The results in Table 1 are obtained directly by substitutions into various equations in the analysis.

Conclusion

The approximations presented here for the spiral trajectory under the perturbations of aerodynamic drag and a low constant tangential thrust are particularly useful for preliminary estimates of propulsion requirements and in mission analyses related to low-altitude orbit around the earth. Attempts should be made to investigate low-thrust spiral trajectories as affected by aerodynamic drag, including the effects of oscillation, which may be useful for guidance design of a satellite.

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Dynamic Stability Testing in a Mach-14 Blowdown Wind Tunnel

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Nomenclature

d = base diameter of cone model, ft
D = coefficient of structural damping moment, ft-lb-sec
I = moment of inertia with respect to pitch axis, slug-ft^2
k = flexure spring constant, ft-lb/rad
M = pitching moment, ft-lb; also Mach number
q = $(\rho/2)V^2$, dynamic pressure, psf
S = $(\pi/4)d^2$, base area, ft^2
T = absolute temperature, °R
V = freestream velocity, fps
α = variation of angle of attack from trim, rad
δ = logarithmic decrement
θ = variation of pitch angle from trim, rad
μ = coefficient of viscosity, slug/sec-ft
ρ = freestream density, slug/ft^3
ω = circular frequency, rad/sec
$(\dot{\ })$ = derivatives with respect to time

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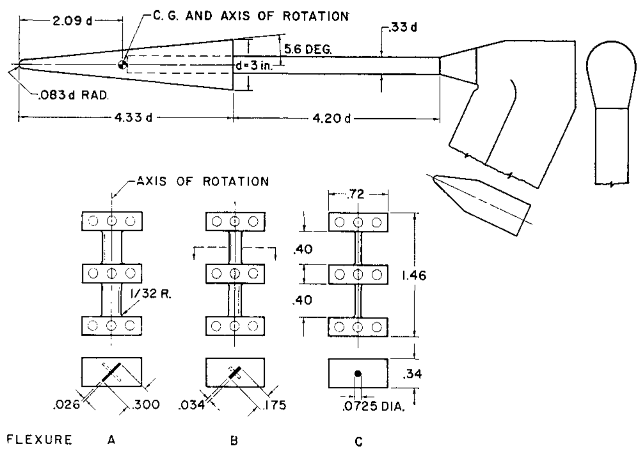

Fig. 1 Model, support system, and flexures used for dynamic stability tests; flexure dimensions in inches.

Subscripts

1 = "wind-on" condition
2 = "wind-off" condition
t = total or isentropic stagnation condition

Definitions

C_m = M/qSd, pitching moment coefficient

C_{m_α} = $(\partial C_m/\partial\alpha)$, static stability derivative

$$C_{m_q} + C_{m_{\dot\alpha}} = \frac{\partial C_m}{\partial[(d/2)(\dot\theta/V)]} + \frac{\partial C_m}{\partial[(d/2)(\dot\alpha/V)]},$$
dynamic stability derivatives

Introduction

THE relatively long-run times of blowdown wind tunnels make this type of hypersonic test facility very attractive for dynamic stability testing. A great number of cycles of oscillation can be observed during one run, even at the lowest frequency of interest. A feasibility study for dynamic stability testing in one of the Aerospace Research Laboratories' hypersonic wind tunnels at Mach 14 is reported here. Use is made of the one-degree-of-freedom, small-amplitude, free-oscillation technique in which the model is connected to the supporting system by an elastic torsion rod (flexure). The data presented refer to zero trim angle of attack only.

Apparatus

For a description of the wind tunnel and its operation, see Refs. 1 and 2. The investigated cone model and the support system are sketched in Fig. 1, which also shows the design of three different beryllium copper flexures. The middle block of the flexure is connected to the stationary sting, and the outer blocks are connected with a mount carrying the model. It is not uncommon in this type of hypersonic testing that the energy dissipation by structural damping moments exceeds the aerodynamic damping by an order of magnitude. The design of the flexures shown in Fig. 1 was intended to reduce the structural damping to a lower level than is usually experienced. The spring constants of 9.10, 7.99, and 8.21 ft-lb/rad were found by static calibration of the flexures A, B, and C, respectively. No change of these values was experienced when an additional load was applied to simulate the effects of a steady lift or drag force. The flexures A and B have the advantage over C of greater structural rigidity in all